Mid-term Exam B.Math III Year (Differential Geometry) 2015

Attempt all questions. Books and notes maybe consulted. Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises in the notes or Pressley's book, which haven't been solved in class must be proved in full if used.)

1. Consider the smooth curve $c : [0, 2\pi] \to \mathbb{R}^3$ defined by $c(t) = (x_1(t), x_2(t), x_3(t))$ where:

$$\begin{aligned} x_1 &= \frac{1}{3}(1 - \sin t) + \frac{1}{\sqrt{3}} \cos t \\ x_2 &= \frac{1}{3}(1 + 2\sin t) \\ x_3 &= \frac{1}{3}(1 - \sin t) - \frac{1}{\sqrt{3}} \cos t \end{aligned}$$

(i): Compute the arc length $L_0^t(c)$. (5 mks)

(ii): Compute the curvature function $k(t)$ of c. (5 mks)

2. Let the smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = ze^x + \sinh(zy) + \sin z$. Show that there exists a neighbourhood $U \subset \mathbb{R}^2$ of $(0, 0)$ and a smooth function $g : U \to \mathbb{R}$ such that (a) $g(0, 0) = 0$ and (b) $f(x, y, g(x, y)) = 0$ for all $(x, y) \in U$. (10 mks)

3. Let $c : [a, b] \to X$ be a smooth curve (parametrised by arc length) on an orientable surface $X \subset \mathbb{R}^3$. Let $\{c'(t), n(t), b(t)\}$ denote the Frenet frame of c, and let ν denote a unit normal field on X. For simplicity of notation, denote $\nu(c(t))$ by $\nu(t)$. Show that:

$$(n(t) \cdot \nu(t))\,(n(t) \cdot \nu'(t)) + (b(t) \cdot \nu(t))\,(b(t) \cdot \nu'(t)) = 0$$

(10 mks)

4. Consider the ellipsoid:

$$X = \left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

(i): Find a coordinate chart (ϕ, V) for X (where V is an open subset of $\mathbb{R}^2$) with the image $\phi(V)$ being the open subset of X given by $U := X \setminus \{(x, 0, z) : x \geq 0\}$. (5 mks)

(ii): Compute the second fundamental form for X on this chart. (5 mks)